1) a) $xy'' + (2-x)y' - 2y = 0$, $y = \int_{c_i} e^{xt} f(t)\,dt$

Substitution $\Rightarrow \int_{c_i} f(t^2-t)\,x e^{xt} + (2t-2) f e^{xt}\,dt = 0 \Rightarrow \left[t(t-1)fe^{xt}\right]_{c_i}$

$+ \int_c \underbrace{\left[2(t-1)f - (2t-1)f - t(t-1)f'\right]}_{\text{set to zero.}} e^{xt} = 0$. Choose $f'/f = \frac{-1}{t(t-1)} = \frac{1}{t} - \frac{1}{t-1}$

$\Rightarrow f = t/(t-1)$ & $y = \int_{c_i} \frac{e^{xt}t}{t-1}\,dt$ if $\left[t^2 e^{xt}\right]_c = 0$. Choose c_1 to be real axis $(-\infty, 0]$ & C_2 to be contour encircling 1.

$y_1 = \int_{-\infty}^{0} \frac{te^{xt}}{t-1}\,dt = \int_0^{\infty} \frac{te^{-xt}}{1+t}\,dt$ & $y_2 = \frac{1}{2\pi i}\oint_{c_2} \frac{e^{xt}t}{t-1}\,dt = e^x$

from Cauchy's integral theorem. At $x=0$ the integral for y_1 diverges & soln finite at $x=0$ is Be^x.

b) $xy'' + (2-x)y' - y = 0$. If $y = 1/x$ we get $x(2/x^3) + (2-x)(-1/x^2) - 1/x = 0$.

For second solution, try $y = uv$, $v = 1/x$ & get $x(u''v + 2u'v' + \cancel{uv''}) + (2-x)(u'v + \cancel{uv'}) - \cancel{uv} = x\cdot\frac{1}{x}w' + 2wx(-1/x^2) + (2-x)w\cdot 1/x = w' - w = 0$

with $w = u'$. So $w = e^x \Rightarrow u = e^x \Rightarrow$ second solution is $y = e^x/x$.

General solution is $\frac{Ae^x + B}{x}$ & the combination finite at $x=0$ is proportional to $\frac{e^x - 1}{x}$.

2) a)  $\int_D P_x + Q_y\,dxdy = \oint_c P\,dy - Q\,dx = \int_0^{\text{Period}} P\frac{dy}{dt} - Q\frac{dx}{dt}\,dt$

$= \int_0^{\text{Period}} PQ - QP\,dt = 0$. So if such a closed contour C represents a periodic solution, then $\int_D = 0 \Rightarrow P_x + Q_y$ must change sign.

b) Dulac's extension is to show $\int (FP)_x + (FQ)_y\,dxdy = 0$

Here $F = e^{ax+by}$, $P = y + x^2$, $Q = -x - y + x^2 + y^2$ &

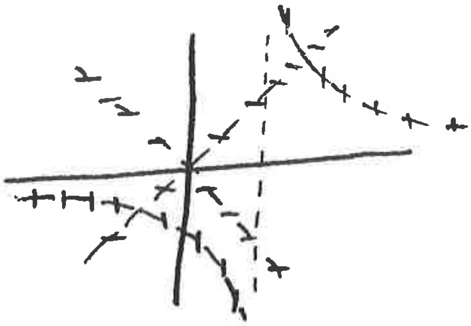
$(FP)_x + (FQ)_y = e^{ax+by}\left[2x + (y+x^2)a + (-1+2y) + (-x-y+x^2+y^2)b\right]$

& if $b = 0$, $a = -2$, this is $e^{ax+by}\left[2x - 2(\cancel{y} + x^2) - 1 + 2\cancel{y}\right]$

$= e^{-2x}(-2)\left[x^2 - x + 1/2\right] = -2e^{-2x}\left[(x - 1/2)^2 + 1/4\right] < 0 \ \forall x, y$. So no periodic solution is possible.

MATH3401 2013

c) i) Vertical nullcline is where $\dot{x} = 0$ ie $-6y + 2xy - 8 = 0 \Rightarrow$ $y(2x-6) = 8$, $y = 4/x-3$. Horizontal nullcline is where $\dot{y} = 0 \Rightarrow y^2 - x^2 = 0$ or $y = \pm x$. (ii) Equilibrium points are where these meet: $y = x$ & $y = 4/x-3 \Rightarrow x^2 - 3x - 4 = (x-4)(x+1)$ $= 0$ if $\underline{x = y = 4}$ or $\underline{x = y = -1}$



OR $y = -x$ & $y = 4/x-3 \Rightarrow x^2 + 3x + 4 = 0$ with no real roots.

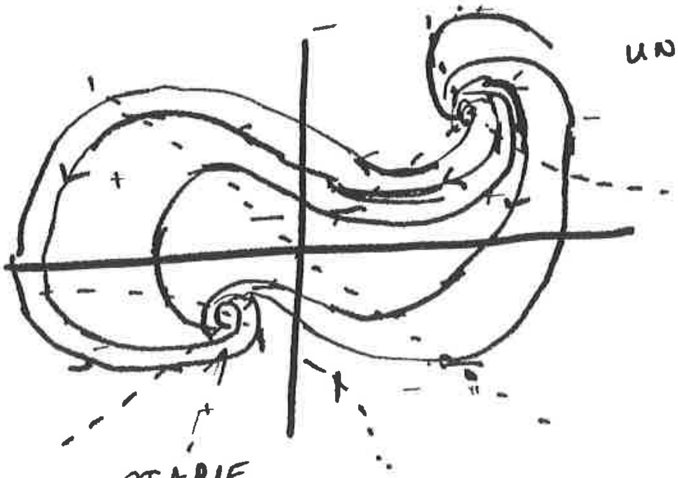
$\underline{\underline{J}} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}$ with $\dot{x} = P$, $\dot{y} = Q$

So $\underline{\underline{J}} = \begin{pmatrix} 2y & -6+2x \\ -2x & 2y \end{pmatrix} = \underset{x=-1}{\begin{pmatrix} -2 & -8 \\ 2 & -2 \end{pmatrix}}$ or $\underset{x=4}{\begin{pmatrix} 8 & 2 \\ -8 & 8 \end{pmatrix}}$

eigenvalues λ at $x = -1$ satisfy $(-2-\lambda)^2 = -16 \Rightarrow -2-\lambda = \pm 4i \Rightarrow \lambda = -2 \pm 4i$ STABLE SPIRAL

$x = 4$ " $(8-\lambda)^2 = -16 \Rightarrow 8 - \lambda = \pm 4i$ $\lambda = 8 \pm 4i$ UNSTABLE SPIRAL

iii) On $y > 0, x = 0$, $\dot{x} < 0$ & $\dot{y} > 0 \Rightarrow dy/dx < 0$



3) a) $\ddot{x} + \varepsilon f(x,\dot{x}) + x = 0 \quad , \varepsilon \ll 1$

with $\quad x = a\sin\theta + \varepsilon x_1 \quad \& \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial\theta}\cdot(1+\varepsilon n_1 + \ldots)$ we find

$$(1+2\varepsilon n_1+\ldots)(-a\sin\theta + \varepsilon\ddot{x}_1\ldots) + \varepsilon f(a\sin\theta+\ldots,(a\cos\theta+\ldots)(1+\ldots))$$
$$+ a\sin\theta + \varepsilon x_1 + \ldots = 0$$

$$\Rightarrow \ddot{x}_1 + x_1 = -f(a\sin\theta, a\cos\theta) + 2an_1\sin\theta.$$

To keep x_1 periodic, we need no component of r.h.s $\propto \cos\theta$ or $\sin\theta$.

$$\Rightarrow -\int_0^{2\pi}\cos\theta f(a\sin\theta, a\cos\theta)\,d\theta + \underbrace{\int_0^{2\pi} 2an_1\cos\theta\sin\theta\,d\theta}_{=0} = 0 \quad \&$$

$$-\int_0^{2\pi}\sin\theta f(a\sin\theta, a\cos\theta)\,d\theta + \underbrace{\int_0^{2\pi} 2an_1\sin^2\theta\,d\theta}_{=2an_1\cdot 2\pi\cdot 1/2 = 2an_1\pi} = 0$$

$$\Rightarrow n_1 = \frac{1}{2\pi a}\int_{-\pi}^{\pi} f(a\sin\theta, a\cos\theta)\sin\theta\,d\theta \quad \& \quad \int_{-\pi}^{\pi} f(a\sin\theta, a\cos\theta)\cos\theta\,d\theta = 0$$

Period $= 2\pi/1+\varepsilon n_1 = 2\pi/1+\varepsilon.0+\ldots = 2\pi(1+o(\varepsilon))$

b) If $f(x,\dot{x}) = \dot{x}h(x)$, then $n_1 = \frac{1}{2\pi a}\int_{-\pi}^{\pi} h(a\sin\theta)\underbrace{a\cos\theta\sin\theta}_{odd}\,d\theta = 0$

if $h(x)$ is even.

c) If $h(x) = \lambda^2 - x^2 - \mu x$ then $\quad \int_{-\pi}^{\pi}(\lambda^2 - a^2\sin^2\theta \overset{-\mu a\sin\theta \text{ odd}}{}) a\underbrace{\cos\theta\cos\theta}_{even}\,d\theta = 0$

$$\Rightarrow a^2 = \lambda^2\int_{-\pi}^{\pi}\cos^2\theta\,d\theta \Big/ \int_{-\pi}^{\pi}\cos^2\theta\sin^2\theta\,d\theta = \lambda^2\frac{2\pi\cdot 1/2}{1/4\cdot 2\pi\cdot 1/2}$$

$= 4\lambda^2$, $\quad$ So $\underline{a = 2\lambda}$ independent of μ.

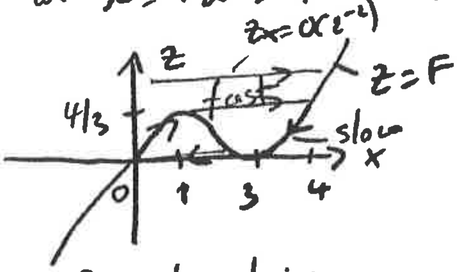
4) Choose $F(x) = \int_0^x f(t)\,dt$, then with $z(t) = \frac{1}{\varepsilon}\dot{x} + F$ (o…

$\dot{z} = \frac{1}{\varepsilon}\ddot{x} + \dot{x}F'(x) = \frac{1}{\varepsilon}\ddot{x} + \dot{x}f = \frac{1}{\varepsilon}(\ddot{x} + \varepsilon\dot{x}f) = -\frac{g}{\varepsilon}$ as $\ddot{x} + \varepsilon f\dot{x} + g = …$

with $\dot{z} = -g/\varepsilon$ & $\dot{x} = \varepsilon(z-F)$, $dz/dx = \frac{1}{\varepsilon^2}\frac{-g}{z-F} = O(\varepsilon^{-2})$, if $z - F = …$

For $f(x) = (x-3)(x-1)$, $F(x) = \frac{1}{3}x^3 - 2x^2 + 3x$, with turning points at $x = 1$ & 3. $F(1) = 4/3$ & the line $z = 4/3$ meets $z = F$ at $x = 1$ & $x = 4$ as $F(x) - 4/3 = \frac{1}{3}(x^3 - 6x^2 + 9x - 4)$

$$= \frac{1}{3}\underbrace{(x^2 - 2x + 1)}_{(x-1)^2}(x-4)$$

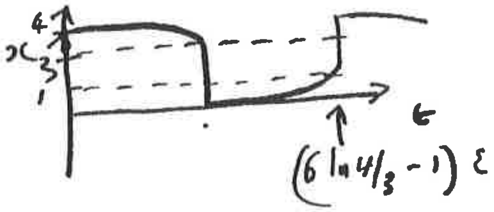


Periodic soln is as shown, min & max are 0 & 4,

Period $\approx 2\int_{4}^{\cancel{4}\,3}\frac{dx}{\dot{x}}$ neglecting contribution from fast part where $\dot{x} = \varepsilon(z-F) \gg 1$

$$= 2\int_{4}^{\cancel{4}\,3}\frac{dz}{dx}\cdot\frac{dt}{dz}\cdot dx = 2\int_{4}^{\cancel{4}\,3}F'(x)\underbrace{\left(\frac{-\varepsilon}{x}\right)}_{\dot{z} = -g/\varepsilon}dx = 2\varepsilon\int_3^4 (x-3)(x-1)\frac{dx}{x}$$

$$= 2\varepsilon\left[\frac{x^2}{2} - 4 + 3\ln x\right]_3^4 = 2\varepsilon\left[\frac{1}{2}(16-9) - 4 + 3\ln 4/3\right] = (6\ln 4/3 - 1)\varepsilon.$$



5) a) $\frac{\ln(1+t)}{t} = \frac{1}{t}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}$ & $\int_0^\infty e^{-xt}\sum_1^\infty(-1)^{n+1}\frac{t^n}{n}\frac{dt}{t}$

$= \int_0^\infty e^{-u}\cdot\sum_1^\infty (-1)^{n+1}\frac{u^{n-1}}{n}\cdot\frac{1}{x^n}\,du = \sum_1^\infty \frac{1}{x^n}\frac{(-1)^{n+1}}{n}\int_0^\infty e^{-u}\cdot u^{n-1}du = \sum_1 \frac{(-1)}{x^n}\dots$

b) $\int_x^\infty e^{-t^4}dt = \int_1^\infty e^{-x^4u^4}{}^{*}\,x\,du = \int_0^\infty e^{-x^4(1+v)^4}x\,dv = e^{-x^4}\int_0^\infty e^{-x^4[(1+v)^4-1]}x\,dv$

$+x^4[(1+v)^4-1] = u \Rightarrow x^4\,4(1+v)^3dv = du \Rightarrow x^4\cdot 4\left(\frac{u}{x^4}+1\right)^{3/4}dv = du$

$= \frac{e^{-x^4}}{4x^3}\int_0^\infty e^{-u}\left(1+\frac{u}{x^4}\right)^{-3/4}du \sim \frac{e^{-x^4}}{4x^3}$

or use $\int_a^b e^{x\varphi(t)}f(t)dt \sim \frac{e^{x\varphi(c)}f(c)}{x|\varphi'(c)|}$ on (*) with $c=1$, $f=x$, $x\to x^4$, $\varphi(u)=u^4$, $\varphi'(c)=4$

c) $\int_a^b e^{ix\varphi(t)}f(t)dt \sim e^{ix\varphi(c)}f(c)\sqrt{\frac{2\pi}{|\varphi''(c)|x}}\,e^{i\,\mathrm{sgn}(\varphi''(c))\pi/4}$, $\varphi'(c)=0$

$\varphi(t)=\cos t$, $c=a=0$, $b=\pi/2$ $\varphi''(c)=-1$ & include a factor ½ as c is at an end point

$\int_0^{\pi/2}e^{ix\cos t}dt \sim \frac{1}{2}e^{ix}\cdot 1\sqrt{\frac{2\pi}{x\cdot 1}}\,e^{-i\pi/4} = \sqrt{\frac{\pi}{2x}}\,e^{i(x-\pi/4)}$

d) $I = \int_0^\infty e^{-x\sinh^2 t}dt = \int_0^\infty e^{-xu^2}\frac{du}{\sqrt{1+u^2}} = \int_0^\infty \frac{e^{-v}}{2\sqrt{x}\sqrt{v}}(1+v/x)^{-1/2}dv$

$u=\sinh t$, $dt = \frac{du}{\sqrt{1+u^2}}$

$v = xu^2$

$dv = 2xu\,du = 2x\sqrt{\frac{v}{x}}\,du = 2\sqrt{x}\sqrt{v}\,du$

$\sim \frac{1}{2}\frac{1}{\sqrt{x}}\left\{\int_0^\infty \frac{e^{-v}}{v^{1/2}}dv + \left(-\frac{1}{2x}\right)\int_0^\infty \sqrt{v}\,e^{-v}dv\dots\right\} = \frac{1}{2\sqrt{x}}\left\{(-\tfrac{1}{2})! - \frac{1}{2x}(\tfrac{1}{2}!)\dots\right\}$

but $(-\frac{1}{2})! = \sqrt{\pi}$ & $(\frac{1}{2})! = \frac{1}{2}(-\frac{1}{2})! = \sqrt{\pi}/2$

so $I \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(1 - \frac{1}{4x}\dots\right)$